

YANG-MILLS FOR QUANTUM HEISENBERG MANIFOLDS

HYUN HO LEE

ABSTRACT. We consider the Yang-Mills problem for a quantum Heisenberg manifold, which is a C^* -algebra defined by the (strict) deformation quantization of the ordinary Heisenberg manifold, in the setting of non-commutative differential geometry following Connes and Rieffel [Co] [Co1].

1. PRELIMINARIES

Classical Yang-Mills theory is concerned with the set of connections (i.e. gauge potentials) on a vector bundle of a smooth manifold. The Yang-Mills functional YM measures the “strength” of the curvature of a connection. The Yang-Mills problem is determining the nature of the set of connections where YM attains its minimum, or more generally the nature of the set of critical points for YM. Since there is a well-developed non-commutative analogue of this setting, we can define a non-commutative Yang-Mills problem as follows [CoRie]: Let (A, G, α) be a C^* -dynamical system, where G is a Lie group. It is said that x in A is C^∞ -vector if and only if $g \rightarrow \alpha_g(x)$ from G to the normed space is of C^∞ . Then $A^\infty = \{x \in A \mid x \text{ is of } C^\infty\}$ is norm dense in A . In this case we call A^∞ the smooth dense subalgebra of A . Since a C^* -algebra with a smooth dense subalgebra is an analogue of a smooth manifold, finitely generated projective A^∞ -modules are the appropriate generalizations of vector bundles over the manifold. By Connes [Co1], a finitely generated projective A^∞ -module Ξ^∞ always exists if there is a finitely generated projective A -module Ξ under G -invariant maps. In addition, an hermitian structure on Ξ^∞ is given by a A^∞ -valued positive definite inner product $\langle \xi, \eta \rangle \in A^\infty$ for $\xi, \eta \in \Xi^\infty$. Let L be the Lie-algebra of (unbounded) derivations of A^∞ given by the representation δ from the Lie-algebra \mathfrak{g} of G where $\delta_X(x) = \lim_{t \rightarrow 0} \frac{1}{t}(\alpha_{\exp(tX)}(x) - x)$ for $X \in \mathfrak{g}$.

Definition 1.1. Given Ξ^∞ , a connection on Ξ^∞ is a linear map $\nabla : \Xi^\infty \rightarrow \Xi^\infty \otimes L^*$ such that, for all $X \in \mathfrak{g}$, $\xi \in \Xi^\infty$ and $x \in A^\infty$ one has

$$\nabla_X(x \cdot \xi) = x \cdot \nabla_X(\xi) + \delta_X(x) \cdot \xi.$$

We shall say that ∇ is compatible with the hermitian metric if and only if

$$\langle \nabla_X(\xi), \eta \rangle + \langle \xi, \nabla_X(\eta) \rangle = \delta_X(\langle \xi, \eta \rangle)$$

for all $\xi, \eta \in \Xi^\infty$, $X \in \mathfrak{g}$.

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Definition 1.2. Let ∇ be a connection on Ξ^∞ , the curvature of ∇ is the element Θ of $\text{End}_{A^\infty}(\Xi^\infty) \otimes \Lambda^2(\mathfrak{g})^*$ given by

$$\Theta_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

for all $X, Y \in \mathfrak{g}$.

If ∇ is compatible with the Hermitian metric, then the values of Θ are the self-adjoint elements of $E = \text{End}_{A^\infty}(\Xi^\infty)$. Since L is playing the role of the tangent space of A^∞ , the analogue of a Riemannian metric on a manifold will be just an ordinary positive inner product on L . With the curvature form in mind, we need the bilinear form on the space of alternating 2-forms with values in E . Then given alternating E -valued 2-forms Φ and Ψ we let

$$\{\Phi, \Psi\}_E = \sum_{i < j} \Phi(Z_i, Z_j) \Psi(Z_i, Z_j),$$

which is an element of E where Z_1, \dots, Z_n is an orthonormal basis of L . Finally, we need an analogue of integration over a manifold, and we need this to be G -invariant. Thus it is appropriate to assume that A^∞ is given a faithful trace τ on A^∞ which is invariant under the action of L i.e. δ -invariant so that $\tau(\delta_X(x)) = 0$ for all $X \in \mathfrak{g}$ and $x \in A^\infty$. One can define an E -valued inner product, \langle, \rangle_E , by

$$\langle \xi, \eta \rangle_E (\zeta) = \xi \cdot \langle \eta, \zeta \rangle$$

Then every element of E will be a finite linear combination of terms of form $\langle \xi, \eta \rangle_E$ so that we can define τ_E by

$$\tau_E(\langle \xi, \eta \rangle_E) = \tau(\langle \xi, \eta \rangle_{A^\infty})$$

Definition 1.3. The Yang-Mills functional YM is defined for a compatible connection ∇ by

$$YM(\nabla) = -\tau_E(\{\Theta_\nabla, \Theta_\nabla\})$$

It is not hard to show that the set of compatible connection $CC(\Xi^\infty)$ is closed under conjugation of a unitary element of E . In fact, if we define $(\gamma_u(\nabla))_X \xi = u(\nabla_X(u^*(\xi)))$ for $u \in UE$, it is easily verified that $\gamma_u(\nabla) \in CC(\Xi^\infty)$. Also, it is verified that

$$\Theta_{\gamma_u(\nabla)}(X, Y) = u \Theta_\nabla(X, Y) u^*$$

for $X, Y \in \mathfrak{g}$, and that

$$\{\Theta_{\gamma_u(\nabla)}, \Theta_{\gamma_u(\nabla)}\} = u \{\Theta_\nabla, \Theta_\nabla\} u^*.$$

It follows that

$$YM(\gamma_u(\nabla)) = YM(\nabla)$$

for every $u \in UE$ and $\nabla \in CC(\Xi^\infty)$. Thus YM is a well-defined functional on the quotient space $CC(\Xi^\infty)/UE$. If $MC(\Xi^\infty)$ denotes the set of compatible connections where YM attains its minimum, we call $MC(\Xi^\infty)/UE$ the moduli space for Ξ^∞ , or more generally $\{\text{the set of critical points}\}/UE$ the moduli space.

Connes and Rieffel [CoRie] studied Yang-Mills for the irrational rotation C^* -algebras which is non-commutative analogue of 2-tori or non-commutative 2-tori and Rieffel extended YM for the higher dimensional non-commutative tori [Rie]. In view of deformation quantization, the higher dimensional non-commutative torus is the example of deformation quantization of d -dimensional torus \mathbb{T}^d . A further aspect of this special deformation quantization is that

the ordinary torus acts on the non-commutative tori as a group of symmetries which is a Lie-group action. According to Rieffel [Rie1], the classical mechanical systems which are studied possess a Lie group action of symmetries acting on the system, and one seeks deformations which are compatible with this Lie group action. Rieffel showed there is a deformation quantization of the classical Heisenberg manifold, namely, non-commutative Heisenberg manifold where the Heisenberg group action is invariant and provided another example of a non-commutative differential manifold [Rie1, Theorem 5.5]. In this short article, we investigate the Yang-Mills problem on a non-commutative Heisenberg manifold as the first small step toward classification of moduli spaces.

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2. MAIN RESULTS

For any positive integer c let S^c be the space of C^∞ functions ϕ on $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ which satisfy

$$(a) \quad \phi(x + k, y, p) = e(ckpy)\phi(x, y, p) \text{ for all } k \in \mathbb{Z}$$

$$(b) \quad \sup_K \|p^k \frac{\partial^{m+n}}{\partial x^m \partial y^n} \phi(x, y, p)\| < \infty \text{ for all } k, m, n \in \mathbb{N} \text{ and any compact set } K \text{ of } \mathbb{R} \times \mathbb{T}$$

We can give S^c a C^* -algebra structure for each $\hbar \in \mathbb{R}$ as follows;

$$(c) \quad \phi * \psi(x, y, p) = \sum_q \phi(x - \hbar(q - p)\mu, y - \hbar(q - p)\nu, q) \psi(x - \hbar q\mu, y - \hbar q\nu, p - q)$$

$$(d) \quad \phi^*(x, y, p) = \overline{\phi}(x, y, -p)$$

with the norm coming from the representation on $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ defined by

$$\phi(f)(x, y, p) = \sum_q \phi(x - \hbar(q - 2p)\mu, y - \hbar(q - 2p)\nu, q) f(x, y, p - q)$$

where μ, ν are non-zero real numbers. $D_{\mu\nu}^{c\hbar}$ will denote the norm completion of S^c . Let G be the Heisenberg group given by

$$(r, s, t) \leftrightarrow \begin{pmatrix} 0 & s & t/c \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix}$$

so that when it is identified with \mathbb{R}^3 the product is given by $(r, s, t)(r', s', t') = (r + r', s + s', t + t' + csr')$. Then we have a canonical action of G on $D_{\mu\nu}^{c\hbar}$ by

$$\alpha_{(r,s,t)}(\phi)(x, y, p) = e(p(t + cs(x - r)))\phi(x - r, y - s, p)$$

which comes from a left action of G on the Heisenberg manifold and $(D_{\mu\nu}^{c\hbar}, G, \alpha)$ is a C^* -dynamical system [Rie1, p557].

From now on, we only consider the case $\hbar = 1$ and thus $D_{\mu\nu}^c$ will denote the corresponding C^* -algebra named by quantum Heisenberg manifold. Thanks to Abadie, we have a different description of $D_{\mu\nu}^c$.

Theorem 2.1. [Ab1, p17] $D_{\mu\nu}^c$ is the closure in the multiplier algebra of $C_0(\mathbb{R} \times \mathbb{T}) \times_\lambda \mathbb{Z}$ of the $*$ -subalgebra D_0 consisting of functions ϕ in $C(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ which have compact support on \mathbb{Z} and satisfy

$$\phi(x + k, y, p) = e(-ckp(y - p\nu))\phi(x, y, p)$$

for all $k, p \in \mathbb{Z}$, and $(x, y) \in \mathbb{R} \times \mathbb{T}$ where $\lambda(x, y) = (x + 2\mu, y + 2\nu)$

There is a faithful trace τ_D on $D_{\mu\nu}^c$ defined for $\phi \in D_{\mu\nu}^c$, by

$$(1) \quad \tau_D(\phi) = \int_{\mathbb{T}^2} \phi(x, y, 0) dx dy$$

The virtue of this alternative definition is the much simpler Morita equivalence picture than the original Morita equivalence found by Rieffel (For the latter, see [Rie2, Theorem 5.5]).

Theorem 2.2. [Ab1, Theorem 2.12] *Let $E_{\mu\nu}^c$ be the closure in the multiplier algebra of $C(\mathbb{R} \times \mathbb{T}) \times_{\sigma} \mathbb{Z}$ of the $*$ -subalgebra E_0 consisting of functions ψ in $C(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ with compact support on \mathbb{Z} and satisfy*

$$\psi(x - 2p\mu, y - 2p\nu, k) = e(ckp(y - p\nu))\psi(x, y, k)$$

for all $k, p \in \mathbb{Z}$, and $(x, y) \in \mathbb{R} \times \mathbb{T}$ where $\sigma(x, y) = (x - 1, y)$. Then $E_{\mu\nu}^c$ and $D_{\mu\nu}^c$ are strong-Morita equivalent via the bimodule $\Xi = C(\mathbb{R} \times \mathbb{T})$.

Corollary 2.3. Ξ is a finitely generated $D_{\mu\nu}^c$ -module and $\text{End}_{D_{\mu\nu}^c}(\Xi)$ is isomorphic to $E_{\mu\nu}^c$ via the map $f \rightarrow f \cdot \psi$.

Proof. Note that both $E_{\mu\nu}^c$ and $D_{\mu\nu}^c$ have identity elements [Ab2, p309]. From the strong Morita equivalence, it is well known that Ξ is a finitely generated and the endomorphism ring $\text{End}_{D_{\mu\nu}^c}(\Xi)$ is equal to $E_{\mu\nu}^c$ (see [Rie2, Proposition 2.1]). \square

For our purpose, we need to recall the action of $D_{\mu\nu}^c$ on Ξ by

$$(\phi \cdot f)(x, y) = \sum_p \phi(x, y, p) f(x - 2p\mu, y - 2p\nu) = \sum_p \phi(x, y, p) (\lambda_{2p} f)(x, y)$$

for $\phi \in D_{\mu\nu}^c$ and $f \in \Xi$
and $D_{\mu\nu}^c$ -valued inner product by

$$\langle f, g \rangle_{D_{\mu\nu}^c}(x, y, p) = \sum_k e(ckp(y - p\nu)) f(x + k, y) \bar{g}(x - 2p\mu + k, y - 2p\nu)$$

for $f, g \in \Xi$.

Accordingly, G act on $D_{\mu\nu}^c$ by

$$(2) \quad \alpha_{(r,s,t)}(\phi)(x, y, p) = e(p(t + cs(x - p\mu - r))) \phi(x - r, y - s, p)$$

for $\phi \in D_{\mu\nu}^c$ and $(D_{\mu\nu}^c, G, \alpha)$ becomes a C^* -dynamical system. Also, we can check that τ is δ -invariant using (1). Since we never work with $D_{\mu\nu}^c$ and Ξ , but only with C^∞ versions, we shall denote the latter by $D_{\mu\nu}^c$ and Ξ . It is easy to see that the Lie algebra of the (parametrized) Heisenberg group has an orthonormal basis consisting of

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have the derivations corresponding to this basis.

Proposition 2.4. *The representation δ of L as a Lie algebra of derivations on $D_{\mu\nu}^c$ is given by*

$$\begin{aligned}\delta_X(\phi)(x, y, p) &= -\frac{\partial}{\partial x}\phi(x, y, p) \\ \delta_Y(\phi)(x, y, p) &= -\frac{\partial}{\partial y}\phi(x, y, p) + 2\pi icp(x - p\mu)\phi(x, y, p) \\ \delta_Z(\phi)(x, y, p) &= 2\pi ip\phi(x, y, p)\end{aligned}$$

Proof. In our case, $\exp(A) = I + \sum_{n=1} A^n/n!$. Using the action of G defined as (2), we can compute $\delta_X(\phi) = \lim_{t \rightarrow 0} \frac{1}{t}(\alpha_{\exp(tX)}(\phi) - \phi)$ for each $X, Y, Z \in \mathfrak{g}$ and $\phi \in D_{\mu\nu}^c$. \square

Thus the smooth dense subalgebra of $D_{\mu\nu}^c$ is the schwartz space of complex valued functions on the $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ satisfying (2.1) and the corresponding finitely generated module is the schwartz space of complex valued functions on the $\mathbb{R} \times \mathbb{T}$. If we let $\{Z_i\}$ be the basis of a Lie algebra \mathfrak{g} , a linear map $\hat{\nabla}^*$ which takes E_g -valued 2 forms to 1-forms is defined by

$$(\hat{\nabla}^*\Omega)(Z_i) = \sum_j [\nabla_{Z_j}, \Omega(Z_i \wedge Z_j)] - \sum_{j < k} c_{jk}^i \Omega(Z_j \wedge Z_k)$$

where c_{jk}^i are the structure constants of \mathfrak{g} for the basis Z_j .

Theorem 2.5. [Rie3, Theorem 1.1] *A compatible connection ∇ is a critical point of YM exactly when it satisfies the Yang-Mills equation $\hat{\nabla}^*(\Theta_\nabla) = 0$*

Now we define a linear map ∇ on Ξ by

$$(3) \quad (\nabla_X f)(x, y) = -\frac{\partial}{\partial x}f(x, y), \quad (\nabla_Y f)(x, y) = -\frac{\partial}{\partial y}f(x, y) + \frac{\pi ci}{2\mu}x^2 f(x, y), \quad (\nabla_Z f)(x, y) = \frac{\pi ix}{\mu}f(x, y)$$

Proposition 2.6. *∇ is a compatible connection.*

Proof. We need to verify that ∇ satisfy the Definition 1.1. It is enough to show that

$$\nabla_W(\phi \cdot f) = \phi \cdot \nabla_W(f) + \delta_W(\phi) \cdot f$$

for $W = X, Y$, and Z .

The first case is obvious.

We note that

$$(\phi \cdot (\nabla_Y f))(x, y) = \sum_p \phi(x, y, p) \frac{\partial}{\partial y}(\lambda_{2p}f)(x, y) + \sum_p \phi(x, y, p) \frac{\pi ci}{2\mu}(x - 2p\mu)^2 f(x - 2p\mu, y - 2p\nu),$$

$$(\delta_Y(\phi) \cdot f)(x, y) = \sum_p \frac{\partial}{\partial y}\phi(x, y, p)(\lambda_{2p}f)(x, y) + \sum_p 2\pi icp(x - p\mu)\phi(x, y, p)f(x - 2p\mu, y - 2p\nu).$$

$$\text{Thus } \phi \cdot (\nabla_Y f) + \delta_Y(\phi) \cdot f = \frac{\partial}{\partial y}(\phi \cdot f) + \frac{\pi ci}{2\mu}x^2(\phi \cdot f) = \nabla_Y(\phi \cdot f).$$

Similarly,

$$(\phi \cdot (\nabla_Z f))(x, y) = \sum_p \phi(x, y, p) \frac{\pi i(x - 2p\mu)}{\mu}(\lambda_{2p}f)(x, y),$$

$$(\delta_Z(\phi) \cdot f)(x, y) = \sum_p 2\pi i p \phi(x, y, p)(\lambda_{2p} f)(x, y)$$

Thus

$$\begin{aligned} (\phi \cdot (\nabla_Z f) + \delta_Z(\phi) \cdot f)(x, y) &= \sum_p \left(\frac{\pi i (x - 2p\mu)}{\mu} + 2\pi i p \right) \phi(x, y, p)(\lambda_{2p} f)(x, y) \\ &= \frac{\pi i x}{\mu} \sum_p \phi(x, y, p)(\lambda_{2p} f)(x, y) \\ &= (\nabla_Z(\phi \cdot f))(x, y) \end{aligned}$$

We also need to check the compatibility. By the product rule of differentiation, it follows that

$$\delta_X(< f, g >_{D_{\mu\nu}^c}) = < \nabla_X f, g >_{D_{\mu\nu}^c} + < f, \nabla_X(g) >_{D_{\mu\nu}^c}$$

Now

$$\begin{aligned} \delta_X(< f, g >)(x, y, p) &= \frac{\partial}{\partial y}(< f, g >_{D_{\mu\nu}^c})(x, y, p) + 2\pi i c p (x - p\mu)(< f, g >_{D_{\mu\nu}^c})(x, y, p) \\ &= \sum_k ((2\pi i c k p) + 2\pi i c p (x - 2p\mu)) e(c k p (y - p\nu)) (\sigma_k f)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \\ &\quad + \sum_k e(c k p (y - p\nu)) \left(\frac{\partial}{\partial y} \sigma_k f \right)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \\ &\quad + \sum_k e(c k p (y - p\nu)) (\sigma_k f)(x, y) \frac{\partial}{\partial y} \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \end{aligned}$$

Note that for each k ,

$$\begin{aligned} \frac{\pi c i}{2\mu} (x + k)^2 + \frac{\overline{\pi c i}}{2\mu} (x - 2p\mu + k)^2 &= \frac{\pi c i}{2\mu} ((x + k)^2 - (x + k - 2p\mu)^2) \\ &= \frac{\pi c i}{2\mu} (2p\mu)(2x - 2p\mu + 2k) \\ &= 2\pi i c p (x - p\mu) + 2\pi i c p k \end{aligned}$$

Thus

$$\begin{aligned} < \nabla_Y f, g > (x, y, p) + < f, \nabla_Y g > (x, y, p) &= \sum_k e(c k p (y - p\nu)) \frac{\partial}{\partial y} (\sigma_k f)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \\ &\quad + \sum_k \frac{\pi c i}{2\mu} (x + k)^2 e(c k p (y - p\nu)) (\sigma_k f)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) + \sum_k \frac{\overline{\pi c i}}{2\mu} (x - 2p\mu + k)^2 e(c k p (y - p\nu)) (\sigma_k f)(x, y) \\ &\quad \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) + \sum_k e(c k p (y - p\nu)) (\sigma_k f)(x, y) \frac{\partial}{\partial y} \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \\ &= \sum_k \left(\frac{\pi c i}{2\mu} (x + k)^2 + \frac{\overline{\pi c i}}{2\mu} (x - 2p\mu + k)^2 \right) e(c k p (y - p\nu)) (\sigma_k f)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) + \sum_k e(c k p (y - p\nu)) (\sigma_k f)(x, y) \\ &\quad \frac{\partial}{\partial y} \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) + \sum_k e(c k p (y - p\nu)) \left(\frac{\partial}{\partial y} \sigma_k f \right)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \\ &= \sum_k (2\pi i c p (x - p\mu) + 2\pi i c p k) e(c k p (y - p\nu)) (\sigma_k f)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) + \sum_k e(c k p (y - p\nu)) (\sigma_k f)(x, y) \\ &\quad \frac{\partial}{\partial y} \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) + \sum_k e(c k p (y - p\nu)) \left(\frac{\partial}{\partial y} \sigma_k f \right)(x, y) \overline{(\lambda_{2p} g)}(\sigma_{-k}(x, y)) \\ &= \delta_Y(< f, g >). \end{aligned}$$

Finally,

$$\begin{aligned}
 & \langle \nabla_Z(f), g \rangle(x, y, p) + \langle f, \nabla_Z(g) \rangle(x, y, p) = \sum_k e(ckp(y - p\nu)) \frac{\pi i(x + k)}{\mu} (\sigma_k f)(x, y) \\
 & \overline{(\lambda_{2p}g)}(\sigma_{-k}(x, y)) + \sum_k e(ckp(y - p\nu)) (\sigma_k f)(x, y) \frac{\pi i(x - 2p\mu + k)}{\mu} \overline{(\lambda_{2p}g)}(\sigma_{-k}(x, y)) \\
 & = \sum_k \left(\frac{\pi i(x + k)}{\mu} + \frac{\pi i(x - 2p\mu + k)}{\mu} \right) e(ckp(y - p\nu)) (\sigma_k f)(x, y) \overline{(\lambda_{2p}g)}(\sigma_{-k}(x, y)) \\
 & = \sum_k 2\pi i p e(ckp(y - p\nu)) (\sigma_k f)(x, y) \overline{(\lambda_{2p}g)}(\sigma_{-k}(x, y)) = \delta_Z(\langle f, g \rangle)(x, y, p). \quad \square
 \end{aligned}$$

Theorem 2.7. *The connection $\nabla : \Xi^\infty \rightarrow \Xi^\infty \otimes L^*$ satisfying (3) is a critical point of Yang-Mills equation for the quantum Heisenberg manifold $D_{\mu\nu}^c$.*

Proof. First, we note that $[X, Y] = -cZ$, $[Y, Z] = 0$, $[Z, X] = 0$. Therefore, $C_{12}^3 = -c$ and all other structure constants are zero. We must show that the Yang-Mills equation $(\hat{\nabla}^*(\Theta_\nabla)) = 0$ for our choice ∇ . Thus it is enough to show that

$$(\hat{\nabla}^*(\Theta_\nabla))(Z_i) = \sum_j [\nabla_{Z_j}, \Theta_\nabla(Z_i \wedge Z_j)] - \sum_{j < k} c_{jk}^i \Theta_\nabla(Z_j \wedge Z_k) = 0$$

for each i .

Let $Z_1 = Z$, $Z_2 = Y$, $Z_3 = X$. We can easily see that $\Theta_\nabla(X, Y) = (-\frac{\pi ci}{\mu} + \frac{\pi ci}{\mu})I_E = 0$,

$\Theta_\nabla(Y, Z) = [\nabla_Y, \nabla_Z] = 0$, $\Theta_\nabla(Z, X) = \frac{\pi i}{\mu} I_E$ where I_E is the identity element of $E = \text{End}_{D_{\mu\nu}^c}(\Xi)$.

Thus

$$\begin{aligned}
 (\hat{\nabla}^*(\Theta_\nabla))(X) &= [\nabla_Y, \Theta_\nabla(X, Y)] + [\nabla_Z, \Theta_\nabla(X, Z)] = 0 \\
 (\hat{\nabla}^*(\Theta_\nabla))(Y) &= [\nabla_X, \Theta_\nabla(Y, X)] = 0 \\
 (\hat{\nabla}^*(\Theta_\nabla))(Z) &= [\nabla_X, \Theta_\nabla(Z, X)] - c_{12}^3 \Theta_\nabla(X, Y) = 0
 \end{aligned}$$

□

Corollary 2.8. *All other critical points of Yang-Mills equation are of the form $\nabla + \gamma$ where $\gamma(Z_i) \in E_{\mu\nu}^c$ such that $\gamma(Z_i)$ is imaginary valued function for each Z_i . In addition, $(\nabla_X + \gamma_X)(f) = \nabla_X(f) + f \cdot \gamma(X)$ where the latter is defined by the action of $E_{\mu\nu}^c$ on Ξ .*

Proof. Suppose ∇' be another critical point. Then $\nabla'_X - \nabla_X$ is a skew-adjoint element of $E = \text{End}_{D_{\mu\nu}^c}(\Xi)$ for each $X \in \mathfrak{g}$. If let $\gamma(X) = \nabla'_X - \nabla_X$, $\nabla'_X = \nabla_X + \gamma(X)$ and $\gamma(X)$ is pure-imaginary valued by Corollary 2.3. □

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: ylee@math.purdue.edu